Abstract: Constructing the density of a bivariate normal random variable requires finding the covariance matrix. It is then needed to assess the standard deviation of both attributes, as well as the correlation coefficients. Usually, subjective data is used as an input, and it is always in an interval form. The paper presents mathematical models to assess correlation coefficients using conditional subjectively elicited (interval) conditional quartiles for two-dimensional normal distributions. Part of the estimates is found by minimization using weighted least square method.

Key words: bivariate normal variables, conditional density, subjective estimates, correlation coefficients
**I. Introduction**

The recognition of the condition of a system/process is a subject of many scientific disciplines, among which pattern recognition (Fukunaga, 1990; Duda, et al., 2001), statistics, econometrics (Hanke & Reitsch, 1991), etc. These approaches are related to and support technical diagnostics, regression analysis, financial portfolio optimisation, application of econometric models to test economic theories, etc. Usually, these approaches give a formal description of the object using some of its main parameters, and find the likelihood that the object is at a certain state.

In most cases, the analysed process/object is described in terms of a multivariate random variable. Then it is required to study the correlation between the parameters. As it is usually the case, complete and adequate empirical information in lacking. That is why subjective estimates are employed.

Due to the partial irrationality of people, their subjective estimates come in an interval form. Therefore, in (Nikolova, et al., 2005) real estimators are referred to as fuzzy rational. This paper presents mathematical models that use interval subjective estimates to find correlation coefficients between the attributes of a bivariate normal variable. Initial studies in that respect were presented in (Nikolova et al., 2010a; Nikolova, et al., 2010b).

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**I. Введение**

Разпознавание состояния объекта или процесса является предметом многих научных подходов, в том числе и паттерн-ре�екognition (Fukunaga, 1990; Duda, et al., 2001), статистика, эконометрика (Hanke & Reitsch, 1991) и т.д. Эти подходы связаны и дополняют технические диагностики, регрессионный анализ, финансовый портфейл оптимизации, применение эконометрических моделей для тестирования экономических теорий, и т.д. Обычно, эти подходы дают формальное описание объекта через его основные параметры и позволяют определить правдоподобие объекта в данном состоянии.

В большинстве случаев, изучение процесса/объекта описывается множеством параметров, т.е. описываемое под множество параметров. Объект или процесс описывается в виде интервального числа. Поэтому в (Nikolova, et al., 2005) реальные оценки называются расширенными. Эта работа представляет математические зависимости и модели, через коэффициенты интервальных оценок можно оценивать степень корреляции между атрибутами двумерной нормальной случайной величины. Основные исследования были опубликованы в (Nikolova et al., 2010a; Nikolova, et al., 2010b).
II. Finding the parameters of a two-dimensional conditional density on subjective estimates

Let’s have a random transposed vector \( \tilde{z} = (x, y)^T \). The transposed vector of mean values is \( \bar{\mu} = (\mu_x, \mu_y)^T \) with coordinates – the mean values of the attributes \( x \) and \( y \). Since \( \tilde{z} \) obeys a two-dimensional normal distribution, then its density is given by:

\[
(1) \quad f(z) = f(x, y) = N(\bar{x}, \bar{\mu}, S) = \frac{1}{2\pi\sqrt{|S|}} \exp\left( -\frac{1}{2} (z - \bar{\mu})^T S^{-1} (z - \bar{\mu}) \right).
\]

Here, \( S \) is the covariance matrix

\[
(2) \quad S = \begin{pmatrix} \sigma_x^2 & r\sigma_x\sigma_y \\ r\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix},
\]

where \( \sigma_x \) is the standard deviation of \( x \), \( \sigma_y \) is the standard deviation of \( y \), and \( r \) is the correlation coefficient between \( x \) and \( y \). The following applies:

\[
(3) \quad \sigma_x \geq 0, \sigma_y \geq 0, \quad r \in [-1; 1],
\]

\[
(4) \quad |S| = \sigma_x^2\sigma_y^2 (1 - r^2),
\]

\[
(5) \quad \sqrt{|S|} = \sigma_x\sigma_y\sqrt{(1 - r^2)},
\]

\[
(6) \quad S^{-1} = \frac{1}{|S|} \begin{pmatrix} \sigma_x^2 & -r\sigma_x\sigma_y \\ -r\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}.
\]

Therefore, in order to construct the two-dimensional density one needs to estimate \( \sigma_x, \sigma_y, r \).

II.1. Estimating standard deviations

The following is true for the normally distributed \( x \):

\[
(7) \quad f(x) = N(x, \mu_x, \sigma_x) = \frac{1}{\sqrt{2\pi\sigma_x}} \exp\left( -\frac{1}{2} \left( \frac{x - \mu_x}{\sigma_x} \right)^2 \right).
\]

A normal random variable \( z \) with mean value of 0 and standard deviation of 1 may be introduced, such that

\[
(8) \quad z = \frac{x - \mu_x}{\sigma_x}.
\]
If $x = x_d \in [x_d^{0.5}, x_d^{0.75}]$ and $\mu_x = x_{0.5} \in [x_{0.5}^{d}, x_{0.5}^{u}]$ are subjectively assessed, then point estimates may be found as:

$$\sigma_x = \frac{x - \mu_x}{z}.$$  \hspace{1cm} (9)

Then $\sigma_x$ may be found as

$$\sigma_x = \frac{x_{0.25}^{mean} - x_{0.5}^{mean}}{z_{0.25}},$$  \hspace{1cm} (10)

$$\sigma_x = \frac{x_{0.75}^{mean} - x_{0.5}^{mean}}{z_{0.75}}.$$  \hspace{1cm} (11)

In the previous formula, $\alpha$ should not be 0.5, because it will lead to uncertainty of type 0/0. If $\alpha$ is close to either 0 or 1, the estimates are too imprecise. A good practice is to use 0.25 and 0.75 as values of $\alpha$:

$$\sigma_{x,1} = \frac{x_{0.25}^{mean} - x_{0.5}^{mean}}{z_{0.25}},$$  \hspace{1cm} (13)

$$\sigma_{x,2} = \frac{x_{0.75}^{mean} - x_{0.5}^{mean}}{z_{0.75}}.$$  \hspace{1cm} (14)

Here, $z_{0.25}$ and $z_{0.75}$ are respectively the 0.25 and 0.75 quantile of a normal distribution $N(0,1)$. If $\sigma_{x,1}$ and $\sigma_{x,2}$ are both estimated, then one obvious possibility is:

$$\sigma_{x,3} = \frac{\sigma_{x,1} + \sigma_{x,2}}{2}.$$  \hspace{1cm} (15)

A better approach is to find $\sigma_x$ using weighted least square (WLS) minimisation (Vuchkov & Stoyanov, 1986; Press et al., 1992):

$$\chi^2_{\sigma,4} = \left( \frac{x_{0.25}^{model} - x_{0.25}^{mean}}{x_{0.25}^{d} - x_{0.25}^{u}} \right)^2 + \left( \frac{x_{0.75}^{model} - x_{0.75}^{mean}}{x_{0.75}^{d} - x_{0.75}^{u}} \right)^2.$$  \hspace{1cm} (16)

It is possible to prove that...
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\( (17) \) \( \sigma_{x,A} = \arg \left( \min_{\sigma_x} \left( x^2_{2,5} \right) \right) = \frac{z_{0.25} \left( x_{0.25}^{\text{mean}} - x_{0.5}^{\text{mean}} \right)}{\left( x_{0.25}^{u} - x_{0.25}^{d} \right)^2} + \frac{z_{0.75} \left( x_{0.75}^{\text{mean}} - x_{0.5}^{\text{mean}} \right)}{\left( x_{0.75}^{u} - x_{0.75}^{d} \right)^2} \).}

**II.2. Estimation of the correlation coefficient**

It is possible to prove that the distribution \( f(\tilde{z}) \) is:

\[
f(\tilde{z}) = f(x, y) = \frac{1}{\sqrt{2\pi\sigma_x\sigma_y}} \exp \left\{ -\frac{1}{2} \left( x - \frac{\mu_x + ry\sigma_x / \sigma_y - r\mu_y\sigma_x / \sigma_y}{\sigma_x\sqrt{1 - r^2}} \right)^2 \right\}.
\]

\[
\cdot \frac{1}{\sqrt{2\pi\sigma_y}} \exp \left\{ -\frac{1}{2} \left( y - \frac{\mu_y}{\sigma_y} \right)^2 \right\} = N(x, \mu_x + ry\sigma_x / \sigma_y - r\mu_y\sigma_x / \sigma_y, \sigma_x\sqrt{1 - r^2}) N(y, \mu_y, \sigma_y) = f(x \mid y) f(y)
\]

Then the conditional density \( f(x \mid y) \) may be presented as

\[
f(x \mid y) = \frac{1}{\sqrt{2\pi\sigma_x\sqrt{1 - r^2}}} \exp \left\{ -\frac{1}{2} \left( x - \frac{\mu_x + ry\sigma_x / \sigma_y - r\mu_y\sigma_x / \sigma_y}{\sigma_x\sqrt{1 - r^2}} \right)^2 \right\} = N(x, \mu_x + ry\sigma_x / \sigma_y - r\mu_y\sigma_x / \sigma_y, \sigma_x\sqrt{1 - r^2}) = N(x, \mu_{x \mid y}, \sigma_{x \mid y}),
\]

\[
(19) \mu_{x \mid y} = \mu_x - r\mu_y\sigma_x / \sigma_y + yr\sigma_x / \sigma_y \quad \sigma_{x \mid y} = \sigma_x\sqrt{1 - r^2}
\]

Then the correlation coefficient \( r \) may be found as:

\[
(20) r = \frac{\mu_{x \mid y} - \mu_x}{\sigma_{x \mid y} - \mu_y\sigma_x / \sigma_y} = \left( \mu_{x \mid y} - \mu_x \right) / \left( \frac{\sigma_{x \mid y} - \mu_y}{\sigma_y} \right).
\]

If \( y = y_a \in [y_a^d, y_a^u] \) and \( \mu_{x \mid y} = (x_{0.5 \mid y_a}) \in \left[ (x_{0.5 \mid y_a})^d, (x_{0.5 \mid y_a})^u \right] \), are subjectively elicited, then:

\[
(21) y_a^{\text{mean}} = \frac{y_a^d + y_a^u}{2},
\]

\[
(22) (x_{0.5 \mid y_a})^{\text{mean}} = \frac{(x_{0.5 \mid y_a})^d + (x_{0.5 \mid y_a})^u}{2},
\]

\[
(23) r = \frac{(x_{0.5 \mid y_a})^{\text{mean}} - \mu_x}{\sigma_x} / \left( \frac{y_a^{\text{mean}} - \mu_y}{\sigma_y} \right).
\]
In the previous formula, $\alpha$ should not be 0.5, because it will lead to uncertainty of type $0/0$. If $\alpha$ is close to either 0 or 1, the estimates are too imprecise. A good practice is to use 0.25 and 0.75 as values of $\alpha$. Then two possible estimates of $r$ follow from (23):

\[
(24) r_1 = \left( \frac{x_{0.5} \mid y_{0.25}}{\sigma_x} - \mu_x \right) \left/ \left( \frac{y_{0.25}^{\text{mean}} - \mu_y}{\sigma_y} \right) \right.,
\]

\[
(25) r_2 = \left( \frac{x_{0.5} \mid y_{0.75}}{\sigma_x} - \mu_x \right) \left/ \left( \frac{y_{0.75}^{\text{mean}} - \mu_y}{\sigma_y} \right) \right.,
\]

Then if both $r_1$ and $r_2$ are estimated, one obvious possibility is

\[
(26) r_3 = \frac{r_1 + r_2}{2}.
\]

A better approach is to find $r$ using weighted least square (WLS) minimisation of:

\[
(27) \chi^2 = \left( \frac{x_{0.5} \mid y_{0.25}^{\text{model}} - (x_{0.5} \mid y_{0.25})^{\text{mean}}}{(x_{0.5} \mid y_{0.25}) - (x_{0.5} \mid y_{0.25})^{\text{mean}}} \right)^2 + \left( \frac{x_{0.5} \mid y_{0.75}^{\text{model}} - (x_{0.5} \mid y_{0.75})^{\text{mean}}}{(x_{0.5} \mid y_{0.75}) - (x_{0.5} \mid y_{0.75})^{\text{mean}}} \right)^2
\]

Then it can be proven that

\[
(28) r_4 = \arg\min_r \left( \chi^2 \right) = \frac{\left( \frac{y_{0.25}^{\text{mean}} - \mu_y}{\sigma_x} \right)^2 - \left( \frac{x_{0.5} \mid y_{0.25}}{\sigma_x} - \mu_x \right)^2}{\left( \frac{y_{0.25}^{\text{mean}} - \mu_y}{\sigma_x} \right)^2 - \left( \frac{x_{0.5} \mid y_{0.25}}{\sigma_x} - \mu_x \right)^2 + \left( \frac{y_{0.75}^{\text{mean}} - \mu_y}{\sigma_x} \right)^2 - \left( \frac{x_{0.5} \mid y_{0.75}}{\sigma_x} - \mu_x \right)^2}
\]

Similar formulae may be derived for $y$ and $x$:

\[
(29) f(\tilde{z}) = f(x, y) = N(\tilde{x}, \tilde{\mu}, S) = N(y, \mu_y + r\sigma_y / \sigma_x - \mu_x \sigma_y / \sigma_x, \sigma_y \sqrt{1-r^2}) N(x, \mu_x, \sigma_x) = f(y \mid x) f(x),
\]
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\[ f(y|x) = \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-r^2}} \exp\left(-\frac{1}{2}\frac{y-(\mu_y + rx\sigma_y / \sigma_x - r\mu_x \sigma_y / \sigma_x)^2}{\sigma_y\sqrt{1-r^2}}\right) \]

\[ = N\left(y, \mu_y + rx\sigma_y / \sigma_x - r\mu_x \sigma_y / \sigma_x, \sigma_y\sqrt{1-r^2}\right) = N\left(y, \mu_{y|x}, \sigma_{y|x}\right) \]

\[
\mu_{y|x} = \mu_y - r\mu_x \sigma_y / \sigma_x + r\sigma_x \sigma_y / \sigma_x, \\
\sigma_{y|x} = \sigma_y\sqrt{1-r^2}.
\]

Then the correlation coefficient \( r \) is

\[
(31) r = \frac{\mu_{y|x} - \mu_y}{x\sigma_y / \sigma_x - \mu_x \sigma_y / \sigma_x} = \frac{\left(\mu_{y|x} - \mu_y\right)}{\left(x - \mu_x\right)}.
\]

If \( x = x_a \in [x_a^d, x_a^u] \) and \( \mu_{y|x} = (y_{0.5} | x_a) \in [y_{0.5} | x_a]^d, (y_{0.5} | x_a)^u \) are subjectively elicited, then:

\[
(32) x_a^{\text{mean}} = \frac{x_a^d + x_a^u}{2},
\]

\[
(33) (y_{0.5} | x_a)^{\text{mean}} = \frac{(y_{0.5} | x_a)^d + (y_{0.5} | x_a)^u}{2},
\]

\[
(34) r = \frac{(y_{0.5} | x_a)^{\text{mean}} - \mu_y}{\sigma_y} / \frac{x_a^{\text{mean}} - \mu_x}{\sigma_x}.
\]

If the values 0.25 and 0.75 are assigned to \( \alpha \), then:

\[
(35) r_5 = \frac{(y_{0.5} | x_{0.25})^{\text{mean}} - \mu_y}{\sigma_y} / \frac{x_{0.25}^{\text{mean}} - \mu_x}{\sigma_x},
\]

\[
(36) r_6 = \frac{(y_{0.5} | x_{0.75})^{\text{mean}} - \mu_y}{\sigma_y} / \frac{x_{0.75}^{\text{mean}} - \mu_x}{\sigma_x}.
\]

If both \( r_5 \) and \( r_6 \) are assessed, then:

\[
(37) r_7 = \frac{r_5 + r_6}{2},
\]

\[
(38) \chi^2_8 = \left[\left(y_{0.5} | x_{0.25}\right)^{\text{model}} - \left(y_{0.5} | x_{0.25}\right)^{\text{mean}}\right]^2 + \left[\left(y_{0.5} | x_{0.75}\right)^{\text{model}} - \left(y_{0.5} | x_{0.75}\right)^{\text{mean}}\right]^2,
\]

Then the correlation coefficient \( r \) is

Тогава корелационния коефициент \( r \) може да се намери като:

\[
(31) r = \frac{\mu_{y|x} - \mu_y}{x\sigma_y / \sigma_x - \mu_x \sigma_y / \sigma_x} = \frac{\left(\mu_{y|x} - \mu_y\right)}{\left(x - \mu_x\right)}.
\]

Ако са налични субективни оценки \( x = x_a \in [x_a^d, x_a^u] \) и \( \mu_{y|x} = (y_{0.5} | x_a) \in [y_{0.5} | x_a]^d, (y_{0.5} | x_a)^u \) то:

\[
(32) x_a^{\text{mean}} = \frac{x_a^d + x_a^u}{2},
\]

\[
(33) (y_{0.5} | x_a)^{\text{mean}} = \frac{(y_{0.5} | x_a)^d + (y_{0.5} | x_a)^u}{2},
\]

\[
(34) r = \frac{(y_{0.5} | x_a)^{\text{mean}} - \mu_y}{\sigma_y} / \frac{x_a^{\text{mean}} - \mu_x}{\sigma_x}.
\]

При стойности 0.25 и 0.75 за \( \alpha \) следват две възможни оценки на \( r \):

\[
(35) r_5 = \frac{(y_{0.5} | x_{0.25})^{\text{mean}} - \mu_y}{\sigma_y} / \frac{x_{0.25}^{\text{mean}} - \mu_x}{\sigma_x},
\]

\[
(36) r_6 = \frac{(y_{0.5} | x_{0.75})^{\text{mean}} - \mu_y}{\sigma_y} / \frac{x_{0.75}^{\text{mean}} - \mu_x}{\sigma_x}.
\]

Ако \( r_5 \) и \( r_6 \) са оценени, тогава:

\[
(37) r_7 = \frac{r_5 + r_6}{2},
\]

\[
(38) \chi^2_8 = \left[\left(y_{0.5} | x_{0.25}\right)^{\text{model}} - \left(y_{0.5} | x_{0.25}\right)^{\text{mean}}\right]^2 + \left[\left(y_{0.5} | x_{0.75}\right)^{\text{model}} - \left(y_{0.5} | x_{0.75}\right)^{\text{mean}}\right]^2,
\]
Let's analyse a bivariate vector $\mathbf{Z} = (x, y)$, whose parameters are related with Bulgarian men: $x$ - height (in cm); $y$ - weight (in kg). Using algorithms for elicitation of subjective quantiles (Nikolova et al., 2004), the following subjective estimates of the quartiles for $x$ and $y$ are found: $x_{0.25} \in [171; 174]$, $x_{0.5} \in [178; 182]$, $x_{0.75} \in [184; 190]$, $y_{0.25} \in [69; 73]$, $y_{0.5} \in [79; 84]$, $y_{0.75} \in [90; 92]$.

If both $r_4$ and $r_8$ are estimated, then a better approach is to find $r$ using WLS minimisation of:

$$
\chi^2 = \chi^2_{4} + \chi^2_{8}.
$$

(40)

In that way, all necessary parameters for the creation of the covariance matrix are estimated.

III. Numerical example

III.1. Input data

Let's analyse a bivariate vector $\mathbf{Z} = (x, y)$, whose parameters are related with Bulgarian men: $x$ - height (in cm); $y$ - weight (in kg). Using algorithms for elicitation of subjective quantiles (Nikolova et al., 2004), the following subjective estimates of the quartiles for $x$ and $y$ are found: $x_{0.25} \in [171; 174]$, $x_{0.5} \in [178; 182]$, $x_{0.75} \in [184; 190]$, $y_{0.25} \in [69; 73]$, $y_{0.5} \in [79; 84]$, $y_{0.75} \in [90; 92]$.

Така са пресметнати всички необходими елементи за създаването на ковариационната матрица.

III. Числов експеримент

III.1. Начални данни

Нека се изследва двумерната случайна величина $\mathbf{Z} = (x, y)$, свързана с два физически параметъра на мъжете в България: $x$ - ръст (в см); $y$ - тегло (в кг). Чрез използване на алгоритми за субективна оценка (Nikolova et al., 2004) са намерени субективни оценки на квартилите на разпределенията на $x$ и $y$: $x_{0.25} \in [171; 174]$, $x_{0.5} \in [178; 182]$, $x_{0.75} \in [184; 190]$, $y_{0.25} \in [69; 73]$, $y_{0.5} \in [79; 84]$, $y_{0.75} \in [90; 92]$. 

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The medians of the height at the two outer quartiles of weight are also subject-
vively elicited: \( (x_{0.5} | y_{0.25}) \in \left[ 172, 173 \right] \), \( (x_{0.5} | y_{0.75}) \in \left[ 187, 189 \right] \), as well as the medians of weight at the two outer quartiles of height: \( (y_{0.5} | x_{0.25}) \in \left[ 70, 71 \right] \), \( (y_{0.5} | x_{0.75}) \in \left[ 89, 90 \right] \).

### III.2. Finding estimates for \( \sigma_x \) and \( \sigma_y \)

It is easy to find the 0.25 and 0.75 quantiles of the normal distribution \( N(0,1) \), as \( z_{0.75} = 0.6745 \), \( z_{0.25} = -0.6745 \) (Hanke & Reitsch, 1991). According to (13) and (14), the first two estimates of \( \sigma_x \) are \( \sigma_{x,1} = 11.1195 \), \( \sigma_{x,2} = 10.3782 \). Then following (15), \( \sigma_{x,3} = 10.7489 \). Using (16) and (17), it is possible to find \( \sigma_{x,4} = 10.9713 \). This estimate is used to construct the distribution of \( x \) on the elicited quartiles (Fig. 1). In the same fashion, the first two estimates of \( \sigma_y \) are \( \sigma_{y,1} = 15.5673 \), \( \sigma_{y,2} = 14.0847 \). Then according to (15), \( \sigma_{y,3} = 14.8260 \). Using (16) and (17), it is possible to find \( \sigma_{y,4} = 15.1111 \). This estimate is used to construct the distribution of \( y \) on the elicited quartiles (Fig. 2). According to (10), the point estimates of the quartiles of \( x \) are: \( x_{0.25}^{\text{mean}} = \frac{171 + 174}{2} = 172.5 \), \( x_{0.5}^{\text{mean}} = \frac{178 + 182}{2} = 180 \), \( x_{0.75}^{\text{mean}} = \frac{184 + 190}{2} = 187 \). According to (21), the point estimates of the quartiles of \( y \) are: \( y_{0.25}^{\text{mean}} = \frac{69 + 73}{2} = 71 \), \( y_{0.5}^{\text{mean}} = \frac{79 + 84}{2} = 81.5 \), \( y_{0.75}^{\text{mean}} = \frac{90 + 92}{2} = 91 \).

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According to (22), the point estimates of the median of $x$ at quartiles $y_{0.25}$ and $y_{0.75}$ are:

$$(x_{0.5} | y_{0.25})_{\text{mean}} = \frac{172 + 173}{2} = 172.5,$$

$$(x_{0.5} | y_{0.75})_{\text{mean}} = \frac{187 + 189}{2} = 188.$$

Then, according to (24) and (25), $r_1=0.5247$, $r_2=0.5074$. Since those estimates are available, then

$$r_3 = \frac{r_1 + r_2}{2} = \frac{0.5247 + 0.5074}{2} = 0.5161.$$

The fourth estimate of $r$ may be found using WLS minimisation of (27), according to (28), as $r_4=0.5145$. The linear dependence between $y$ and $x$ at $r_4=0.5145$ is given in fig. 3.

In the same fashion, according to (33), the point estimates of the median of the weight $y$ at the quartiles $x_{0.25}$ and $x_{0.75}$ are:

$$(y_{0.5} | x_{0.25})_{\text{mean}} = \frac{70 + 71}{2} = 70.5,$$

$$(y_{0.5} | x_{0.25})_{\text{mean}} = \frac{89 + 90}{2} = 89.5.$$

According to (22), the point estimates of the median of $x$ at quartiles $y_{0.25}$ and $y_{0.75}$ are:

$$(x_{0.5} | y_{0.25})_{\text{mean}} = \frac{172 + 173}{2} = 172.5,$$

$$(x_{0.5} | y_{0.75})_{\text{mean}} = \frac{187 + 189}{2} = 188.$$

Then, according to (24) and (25), $r_1=0.5247$, $r_2=0.5074$. Since those estimates are available, then

$$r_3 = \frac{r_1 + r_2}{2} = \frac{0.5247 + 0.5074}{2} = 0.5161.$$

The fourth estimate of $r$ may be found using WLS minimisation of (27), according to (28), as $r_4=0.5145$. The linear dependence between $y$ and $x$ at $r_4=0.5145$ is given in fig. 3.

In the same fashion, according to (33), the point estimates of the median of the weight $y$ at the quartiles $x_{0.25}$ and $x_{0.75}$ are:

$$(y_{0.5} | x_{0.25})_{\text{mean}} = \frac{70 + 71}{2} = 70.5,$$

$$(y_{0.5} | x_{0.25})_{\text{mean}} = \frac{89 + 90}{2} = 89.5.$$
Then, according to (35) and (36), $r_5=0.5808$, $r_7=0.5186$. Since those estimates are available, then $r_7 = \frac{r_5 + r_6}{2} = \frac{0.5808 + 0.5186}{2} = 0.5497$. The fourth estimate of $r$ may be found using WLS minimisation of (38), according to (39), as $r_6=0.5519$. The linear dependence between $x$ and $y$ at $r_8=0.0.5519$ is given in fig. 4. If both $r_4$ and $r_8$ are available, then using (40) and (41), $r_9=0.5411$. This is employed to construct the linear dependencies on fig. 5.

Then, according to (35) and (36), $r_5=0.5808$, $r_7=0.5186$. При наличие на тези две оценки, може да се намира $r_7 = \frac{r_5 + r_6}{2} = \frac{0.5808 + 0.5186}{2} = 0.5497$. Чрез WLS минимизация на (38), съгласно (39), се намира $r_6=0.5519$. Линейната връзка между $x$ и $y$ при $r_8=0.0.5519$ е дадена на фиг. 4. При наличие на стойности за $r_4$ и $r_8$, то чрез (40) и (41) се намира, че $r_9=0.5411$. Тази оценка е използвана за построяване на двете линейни зависимости на фиг. 5.

### IV. Conclusions

The paper presented the task to estimate the correlation coefficients for a bivariate normal random variable. Subjective interval probability estimates were used to find those estimates. Four estimates for the standard deviation, as well as nine estimates of $r$ were proposed. Once those estimates were available, it was possible to find the covariance matrix and construct the density of the variable.
The elaborated procedures are realised in original software in Matlab R2010a. The results from the paper may be applied in different analyses in economic and financial processes, technical diagnostics, etc. The formal proof of the mathematical dependencies is the subject of future publications.

All algorithms for subjective elicitation of the elements of the covariance matrix often generate negative estimates, which does not comply with its classical structure. Such a matrix is referred to as fictitious (Tenekedjiev et al., 2000) and cannot be further used, especially in pattern recognition tasks. That is why (Tenekedjiev et al., 2000; Nikolova, et al., 2010a) present algorithms to transform fictitious covariance matrices into classical ones.

Reference/ Литература


1 This paper is developed and financially supported by project DVU01/0031 “INPORT” and by project TK01/045: “Intelligent systems for diagnostics and decision making in technological processes” of the National Science Fund.

1 Трудът е разработен и финансово подпомогнат в проект DVU01/0031 “ИНПОРТ” и в проект TK01/045: „Интелигентни системи за диагностика и вземане на решения в технологични процеси” на Фонд „Научни изследвания“.